NON UNIFORMLY HYPERBOLIC ATTRACTORS DERIVED FROM THE STANDARD MAP

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Abstract. We prove that the system resulting of coupling the standard map with a fast hyperbolic system is robustly non-uniformly hyperbolic.

1. Introduction and Statement of the Results

The theory of uniform hyperbolicity, or "Axiom A" diffeomorphism is presently almost complete. It enjoys of various examples [Sma67] and implies strong mixing ergodic properties [Bow08]. However, this theory does not include many chaotic attractors, especially due to obstructions which are topological (existence of the stable plane field on the attraction basin) or geometrical (robust tangencies).

To include more examples, Y. Pesin has introduced the very general concept of non uniform hyperbolicity. A diffeomorphism f of a manifold M is *non uniformly hyperbolic* (NUH) with respect to an invariant probability μ if for μ-almost every $x \in M$, it holds for every unit vector $v \in T_xM$:

$$\lim_{n\to\infty} \frac{1}{n} \log \|df^n(v)\| \neq 0 \tag{1}$$

Actually, by Oseledec Theorem, the above limit exists for μ a.e. x; T_xM is the direct sum of df-invariant subspaces, and on each of these subspaces the above limit converges to an f-invariant value called the *Lyapunov exponent*. Hence f is non uniformly hyperbolic w.r.t. μ iff its Lyapunov exponents are non zero.

A case of main interest is when the measure μ is *physical*, that is when the basin of μ (formed by the points x for which the Birkhoff sum $\frac{1}{n}\sum_k \delta_{f^k(x)}$ converges weakly to μ) has positive Lebesgue measure. We say then that f has a *NUH attractor*.

Such attractors have many properties [Pes77], however their ergodic properties depend on additional hypothesis. In order, to complete such a theory, more examples are needed. Let us list our favorite examples of abundant non uniformly hyperbolic attractors:

(*i*) Jakobson Theorem [Jak81]: for a Lebesgue positive set of parameters $c \in \mathbb{R}$, the quadratic family $x^2 + c$ has a NUH attractor.

(*ii*) Rees Theorem [Ree86]: for a Lebesgue positive set of rational functions (of degree $d \geq 2$), there exists a NUH attractor.

(*iii*) Benedicks-Carleson Theorem [BC91],[BY93],[Ber]: for a Lebesgue positive set of parameter (a, b) with b small, the Hénon map

$$h_{ab}(x, y) = (x^2 + y + a, -bx)$$

has a NUH attractor.

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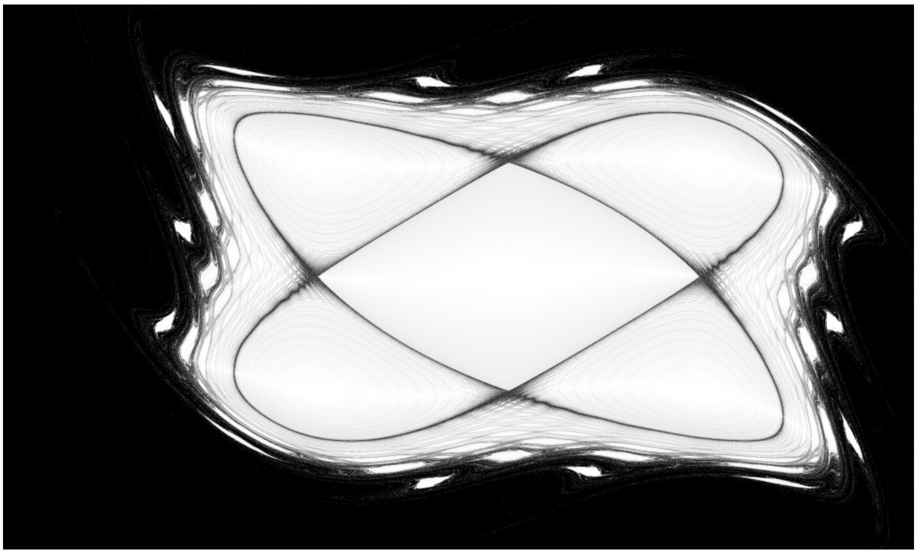

FIGURE 1. Lyapunov exponent for the parameter $r = -0.364$ of the standard map, white regions are expected to be KAM islands whereas the black regions are possibly the homoclinic web of a NUH attractor.

In all these examples, the (real or complex) dimension of the attractor is one or close to 1, and they are *abundant*: they hold for a Lebesgue positive set of parameters (of generic families).

An interesting candidate to this list is the Chirikov-Taylor Standard map family $(S_r)_{r\in\mathbb{R}}$, which is formed by conservative maps of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$S_r(x, y) = (2x - y + r\sin(x), x).$$

It is conjectured that for a Lebesgue positive set of large parameters r, the map S_r has a non uniformly hyperbolic attractor. For such parameters the Hausdorff dimension of the attractor should be close to 2.

This conjecture is very hard: the dynamics of the standard map is not understood for any parameter $r \neq 0$. On the other hand, the important negative result of P. Duarte [Dua94] states that for a residual set of parameters in $r \geq r_0$, infinitely many KAM islands coexist (see fig. 1).

Another way to construct examples of NUH attractors is to work with dynamics which (locally) fiber over a uniformly hyperbolic one. Such a techniques have been used notably by [Shu71], [Via97], and [SW00a] to produce new examples which are robustly non uniformly hyperbolic.

Definition 1.1. *A map f of a compact manifold M is $\mathcal{C}^s$-*robustly non-uniformly hyperbolic *if there exists a $\mathcal{C}^s$-neighborhood U of f such that every map $g \in U$ has a NUH attractor.*

The above list of examples is almost exhaustive. There are very few NUH attractors which have both non uniformly contracting and expanding directions (and which are abundant). The sophisticated techniques used by Benedicks and Carleson to study the Hénon attractor are not completely satisfactory for a general theory since they need a small determinant b.

It has been recently given a new example [AV10]: for most symplectic perturbations of a non-hyperbolic ergodic automorphism, there exists a NUH attractor. The techniques developed there are also suitable to deal with some conservative cases, pushing forward the method developed in [SW00a] for volume preserving diffeomorphisms.

In this work we show the existence of a robust NUH attractor for a map which is volume preserving and which has both non uniformly contracting and expanding directions.

Let $A \in SL_2(\mathbb{Z})$ be a hyperbolic matrix with eigenvalues $\lambda < 1 < 1/\lambda$. Consider the manifold $M = \mathbb{T}^2 \times \mathbb{T}^2$ with coordinates $m = (x, y, z, w)$, and the analytic diffeomorphism $f_N : M \to M$ given by

$$f_N(m) = (S_N(x,y) + A^N(z,w), A^{2N}(z,w)) \quad \text{where } N \geq 0.$$

Theorem 1.1. *There exists N_0 and $c > 0$ such that for every $N \geq N_0$, the map f_N satisfies for Lebesgue a.e. $m \in \mathbb{T}^2 \times \mathbb{T}^2$ and every unit vector $v \in \mathbb{R}^4$:*

$$\lim_{n\to\infty} \Big|\frac{1}{n} \log \|Tf_N^n(v)\|\Big| > c \log N.$$

Moreover the same holds for every conservative map in a $\mathcal{C}^2$-neighborhood of f_N.

A direct consequence of Pesin's ergodic decomposition Theorem for Hyperbolic Measures [Pes77] is:

Corollary 1.1. *There exists N_0 such that for every $N \geq N_0$, in a neighbourhood U of f in the space of $\mathcal{C}^2$-conservative maps, every map has a NUH attractor.*

A similar result was announced in [Bar00]. We hope that this new example will help be helpful to study the standard map, notably by wondering about the ergodicity and mixing property of the Lebesgue measure with the dynamics f_N (in the last section we will see that for most of the volume preserving perturbations of f_N, the Lebesgue measure is ergodic).

In the last section, we will give a larger class of maps on which the presented proof seems to work with minor changes (noted there), in order to obtain the same conclusions as for $(f_N)_N$.

There is a physical interpretation of the coupling of S_N with such an Anosov, the dynamics on the basis would model the microscopic forces which perturb the macroscopic scale endowed with the dynamics S_N.

2. Preliminaries

Along the proof N will be supposed larger and larger in order to satisfy several induction free conditions.

In this section we collect some basic facts about the dynamical properties of f_N and its inverse. Let us begin by noting the following symmetry property.

Lemma 2.1. *The map f_N^{-1} is conjugated to the map*

$$m = (x, y, z, w) \mapsto (S_N(x,y) + A^{-N}(z,w), A^{-2N}(z,w)).$$

Proof. Consider the involutions

$$R(x,y) = (y,x) \quad \text{and} \quad J(x,y) = (x, 2x - y + N \sin x)$$

and observe that $R \circ J = S_N$ hence $S_N^{-1} = J \circ R$ (this is the *reversibility* of the standard map). To seek the inverse of f_N, we put

$$(a,b,c,d) = (S_N(x,y) + A^N(z,w), A^{2N}(z,w)).$$

Thus $(z,w) = A^{-2N}(c,d)$, also

$$\begin{aligned}(x,y) = S_N^{-1}((a,b) - A^N(z,w)) &= S_N^{-1}((a,b) - A^{-N}(c,d)) \\ &= J(R(a,b) - R \circ A^{-N}(c,d)) = J \circ R(a,b) + R \circ A^{-N}(c,d)\end{aligned}$$

since $J((i,j) - (0,k)) = J(i,j) + (0,k)$. We conclude

$$f_N^{-1}(a,b,c,d) = (J \circ R(a,b) + R \circ A^{-N}(c,d), A^{-2N}(c,d)). \tag{2}$$

Finally, if $\hat{R}$ is the involution $\hat{R}(a,b,c,d) = (R(a,b),c,d)$, we obtain

$$\hat{R} \circ f_N^{-1} \circ \hat{R}(a,b,c,d) = (S_N(a,b) + A^{-N}(c,d), A^{-2N}(c,d)). \tag{3}$$

□

This Lemma allows us to establish simultaneously properties for f_N and f_N^{-1}. For example, to prove that f_N is NUH it suffices to show that it has two positive exponents.

As mentioned, the map f_N will be proved to be partially hyperbolic. We recall here this notion and the important stable manifold theorem.

Definition 2.1. *Let M be a closed Riemannian manifold. A $\mathcal{C}^1$ diffeomorphism $f: M \to M$ is* partially hyperbolic *if there exists a continuous splitting of the tangent bundle of the form*

$$TM = E^u \oplus E^c \oplus E^s$$

where both bundles E^s, E^u have positive dimension, and such that

(1) *All bundles E^u, E^s, E^c are df-invariant.*
(2) *For every $m \in M$, for every unitary vector $v^\sigma \in E^\sigma_m, \sigma = s,c,u$,*

$$\|d_m f(v^s)\| < 1 < \|d_m f(v^u)\|$$
$$\|d_m f(v^s)\| < \|d_m f(v^c)\| < \|d_m f(v^u)\|$$

The bundles E^s, E^u, E^c are called the *stable, unstable* and *center* bundle respectively.

Theorem 2.1 (Stable Manifold Theorem)**.** *If $f \in Diff^r(M)$ is partially hyperbolic then the bundles E^s, E^u are (uniquely) integrable to continuous foliations $\mathcal{F}^s = \{W^s(m)\}, \mathcal{F}^u = \{W^u(m)\}$ resp. called the* stable *and the* unstable *foliations, whose leaves are $\mathcal{C}^r$ immersed submanifolds.*

See [Pes04] and the references therein for an account of partial hyperbolicity and the proof of the previous results. Let us come back to the study of f_N.

Consider a normal eigenbasis $\{e^s, e^u\}$ of A corresponding to the respective eigenvalues $\lambda < 1 < \frac{1}{\lambda} =: \mu$. Put $E^u_A = span\{e^u\}$ and $E^s_A = span\{e^s\}$. We also consider the following vector subspace $\mathbb{R}^4$:

$$E^u_0 = \{0\} \times E^u_A, \quad E^s_0 = \{0\} \times E^s_A, \quad E^c = \mathbb{R}^2 \times \{0\}.$$

For each $m \in M, p \in \mathbb{T}^2$ we identify $T_m M \equiv \mathbb{R}^4,\ T_p\mathbb{T}^2 \equiv \mathbb{R}^2$.

Proposition 2.1. *If N is sufficiently large the map f_N is partially hyperbolic with center bundle E^c. Moreover, if E^u_N denotes the corresponding unstable bundle of f_N, then*

$$\diamondsuit \qquad |sin\angle(E^u_N, (\lambda^N e^u, e^u))| \leq \lambda^{2N}/2.$$

Moreover for every $u \in E^u_N$:

$$\|df_N(u)\| = \mu^{2N}\|u\|.$$

A similar statement holds for the stable bundle E^s_N.

A consequence of Prop. 2.1,$\diamondsuit$ is, with $P_x : \mathbb{R}^4 \to \mathbb{R}$ the first coordinate projection:

Corollary 2.1. *For all $m \in M$ and unit vector $v = (v_x, v_y, v_z, v_w) \in E^u_N(m)$, it holds:*

$$\lambda^N(1-\lambda^N)\,\|P_x(e^u)\| \leq |v_x| \leq \lambda^N(1+\lambda^N)\,\|P_x(e^u)\|$$

and $\|P_x(e^u)\| > 0$.

Similar statements hold for the stable bundle E^s_N.

Proof of Proposition 2.1. The differential of f_N at a point $m = (x, y, z, w) \in M$ is given by

$$d_m f_N = \begin{pmatrix} d_{(x,y)} S_N & A^N \\ 0 & A^{2N} \end{pmatrix}$$

From here we deduce that E^c is a df-invariant bundle, and furthermore the action on a vector $v = (v_x, v_y, 0, 0) \in E^c_m$ is given by

$$d_m f_N(v_x, v_y, 0, 0) = (d_{(x,y)} S_N(v_x, v_y), 0, 0) \tag{4}$$

In particular (by reversibility of the standard map), we have

$$\frac{1}{2N} \|v\| \le \|d_m f_N(v)\| \le 2N \|v\| \tag{5}$$

Let us study the unstable direction. For $m \in M$, consider:

$$\alpha_m := \lambda^N \sum_{k=-\infty}^{0} \lambda^{2Nk} d_{(x_{-1},y_{-1})} S_N \circ \cdots \circ d_{(x_{-k},y_{-k})} S_N(e^u),$$

with $(x_j, y_j, z_j, w_j) := f_N^j(m)$.

We remark that:

$$E^u_N(m) := \mathbb{R} \cdot (\alpha_m, e^u)$$

is a continuous line field on M, since $\|\lambda^{2N} dS_N\| < \lambda^N/2 < 1$. Moreover:

$$d_m f_N(\alpha_m, e^u) = (d_{(x_0,y_0)} S_N \alpha_m + \mu^N e^u, \mu^{2N} e^u) = \mu^{2N}(\alpha_{f_N(m)}, e^u)$$

Indeed $\frac{d_{(x_0,y_0)} S_N \alpha_m + \mu^N e^u}{\mu^{2N}} = \lambda^{2N} \left(d_{(x_0,y_0)} S_N \alpha_m + \mu^N e^u\right)$ is equal to

$$\begin{aligned}
&\lambda^N e^u + \lambda^{2N} d_{(x_0,y_0)} S_N(\lambda^N \sum_{k=-\infty}^{0} \lambda^{2Nk} d_{(x_{-1},y_{-1})} S_N \circ \cdots \circ d_{(x_{-k},y_{-k})} S_N(e^u) \\
&= \lambda^N e^u + \lambda^N \sum_{k=-\infty}^{0} \lambda^{2N(k+1)} d_{(x_0,y_0)} S_N \circ \cdots \circ d_{(x_{-k},y_{-k})} S_N(e^u) \\
&= \alpha_{f_N(m)}
\end{aligned}$$

□

From now on we work with N for which the previous Proposition holds.

Remark 2.1. *Observe that the homological action* $(S_N)_* = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} : H_1(\mathbb{T}^2; \mathbb{R}) \to H_1(\mathbb{T}^2; \mathbb{R})$ *of* S_N *is unipotent, and so the one of* f_N *is not hyperbolic. Hence by* [Fra69] *the map* f_N *is not uniformly hyperbolic.*

Let us identify E^c to $\mathbb{R}^2$. By equation (4), we see that $d_m f_N | E^c : \mathbb{R}^2 \to \mathbb{R}^2$ coincides with $d_{(x,y)} S_N$. From now on we will identify:

$$df_N | E^c = dS_N.$$

Note that $\|d^2 S_N\| \le N$, hence using (5) we conclude

$$\frac{\|u\|}{2N} \le \|df_N(u)\| \le 2N \|u\|, \quad \forall u \in E^c, \quad \|d^2 f_N\| \le N \tag{6}$$

In general the center bundle of a partially hyperbolic system is not integrable (see [HHU10]). However, in our case by construction E^c integrates to the smooth fibration by tori $(\mathbb{T}^2\times\{w\})_{w\in\mathbb{T}^2}$.

Corollary 2.2 (Hirsch-Pugh-Shub [HPS77])**.** *There exists a neighbourhood $U \subset Diff^2(M)$ of f_N such that every $g \in U$ is partially hyperbolic with center bundle E^c_g integrable to a continuous fibration by $\mathcal{C}^2$-tori depending continuously on the base point.*

3. Central Exponents.

We will first prove non uniform hyperbolicity of the map f_N, and later in Section 6 we will indicate the necessary modifications to our arguments to deal with perturbations. To make the notation less cumbersome we write $f = f_N$.

In the unstable direction f has one positive Lyapunov exponent with respect to the Lebesgue measure. We are now seeking another positive exponent in the center direction. Let us introduce a few notions to perform this.

Definition 3.1. *A u-curve for f is a curve $\gamma = (\gamma_x, \gamma_y, \gamma_y, \gamma_w) : [0, 2\pi] \to M$, tangent to E^u and such that*

$$|\frac{d\gamma_x}{dt}(t)| = 1 \quad \forall t \in [0, 2\pi].$$

As every u-curve γ is a segment of an unstable leaf on which the action of df_N is the mere multiplication by μ^{2N} (by Prop. 2.1), we have:

$$|\frac{d(f^k \circ \gamma)_x}{dt}(t)| = \mu^{2kN}, \quad \forall t \in [0, 2\pi],\ \forall k \geq 0. \tag{7}$$

where $(f^k \circ \gamma)_x \in \mathbb{R}/2\pi\mathbb{Z}$ is the first coordinate of the curve $f^k \circ \gamma$. Hence, with $[\mu^{2kN}]$ the integer part of μ^{2kN}, the curve $f^k \circ \gamma$ can be written as a concatenation

$$f^k \circ \gamma = \gamma_1^k * \cdots \gamma^k_{[\mu^{2kN}]} * \gamma^k_{[\mu^{2kN}]+1}$$

where γ_j^k $j = 1, \dots, [\mu^{2kN}]$ are u-curves and $\gamma^k_{[\mu^{2kN}]+1}$ is a segment of u-curve.

To prepare the perturbative case, we look at 1/2-Hölder vector fields on u-curves.

Definition 3.2. *An* adapted field (γ, X) *is the pair of a u-curve γ and of a unit vector field X satisfying:*

1. *X is tangent to the center direction.*
2. *X is $(C_X, 1/2)$-Hölder along γ, meaning*
$$\forall m, m' \in \gamma, \quad \|X_m - X_{m'}\| \leq C_X d(m, m')^{1/2}$$
with $C_X < 10N^2\lambda^N$ and $d(m, y)$ the distance along γ.

From Corollary 2.1 one deduces that the length of a u-curve is bounded by

$$\frac{\mu^N}{(1-\lambda^N)\,\|P_x(e^u)\|},$$

hence:

Corollary 3.1. *If N is sufficiently large then for every adapted field (γ, X), for all $m, m' \in \gamma$:*

$$\|X_m - X_{m'}\| < \lambda^{N/3}.$$

Proof. Indeed,

$$\|X_m - X_{m'}\| < 10N^2\lambda^N \left(\frac{\mu^N}{(1-\lambda^N)\,\|P_x(e^u)\|}\right)^{1/2} < \lambda^{N/3}.$$

□

Fix and adapted field (γ, X) and let $Y^k := \frac{(f^k)_*X}{\|(f^k)_*X\|}$. Denote by $d\gamma$ the Lebesgue measure induced on[1] γ and by $|\gamma|$ the length of γ. We compute

$$\begin{aligned} I_n^{\gamma,X} &:= \frac{1}{|\gamma|}\int_\gamma \log \left\|d_m f^k X_m\right\| d\gamma \\ &= \sum_{k=0}^{n-1} \frac{1}{|\gamma|}\int_\gamma \log \left\|d_{f^k m} f(Y^k \circ f^k(m))\right\| d\gamma \\ &= \sum_{k=0}^{n-1} \frac{1}{\mu^{2Nk}|\gamma|}\int_{f^k\circ\gamma} \log \left\|d_m f(Y^k)\right\| d(f^k\circ\gamma), \end{aligned}$$

where in the last equality we have used a change of variables together with the fact that $\left\|\frac{d(f^k\circ\gamma)}{dt}(t)\right\| = \mu^{2Nk}\left\|\frac{d\gamma}{dt}(t)\right\|$.

We conclude

$$I_n^{\gamma,X} = \sum_{k=0}^{n-1} \frac{1}{\mu^{2Nk}|\gamma|}\left(\sum_{j=0}^{[\mu^{2kN}]}\int_{\gamma_j^k} \log\left\|d_m f(Y^k)\right\| d\gamma_j^k + \int_{\gamma^k_{[\mu^{2kN}]+1}} \log\left\|d_m f(Y^k)\right\| d\gamma^k_{[\mu^{2kN}]+1}\right) \tag{8}$$

A crucial point is that all pairs $(\gamma_j^k, Y^k|\gamma_j^k)$ for $1 \le j \le [\mu^{2Nk}]$ are adapted fields.

Lemma 3.1. *For N sufficiently large, for every adapted field (γ, X), for every $k \ge 1$, for every $1 \le j \le [\mu^{2N}]$, the pair $(\gamma_j^k, Y^k|\gamma_j^k)$ is an adapted field.*

Proof. The only non trivial assertion is that Y is $(C_Y, 1/2)$-Hölder with $C_Y < 10N^2\lambda^N$. Let us do the proof for $k = 1$. We write $Y := Y^1$. Observe that

$$\forall m, m' \in M, \quad \|d_m f(X_m) - d_{m'} f(X_m)\| \le N d(m, m').$$

and since for every $m \in M$, $\|d_m f(X_m)\| < 2N$ (by equation (6)), it holds

$$\forall m, m' \in M, \quad \|d_m f(X_m) - d_{m'} f(X_m)\| \le 4N d(m, m')^{1/2}.$$

We compute for $m, m' \in \gamma_j^1$ using the triangular inequality:

$$\begin{aligned} \|Y_m - Y_{m'}\| &= \frac{1}{\|f_*X_m\|\,\|f_*X_{m'}\|}\Big\| \|f_*X_{m'}\|\, f_*X_m - \|f_*X_m\|\, f_*X_{m'}\Big\| \\ &\le \frac{2}{\|f_*X_m\|}\|f_*X_m - f_*X_{m'}\| \\ &= \frac{2}{\|f_*X_m\|}\left\|d_{f^{-1}m} f(X_{f^{-1}m}) - d_{f^{-1}m'} f(X_{f^{-1}m'})\right\| \\ &\le \frac{2}{\|f_*X_m\|}\Big(\|df|_{E^c}\|\, C_X d(f^{-1}m, f^{-1}m')^{1/2} \\ &\qquad + \left\|d_{f^{-1}m} f(X_{f^{-1}m'}) - d_{f^{-1}m'} f(X_{f^{-1}m'})\right\|\Big) \end{aligned}$$

Using equation (3) and since $\|df|_{E^c}\| \le 2N$, we finally get

$$\|Y_m - Y_{m'}\| \le 8N^2(\lambda^N C_X + \lambda^N) d(m, m')^{1/2} < 10N^2\lambda^N d(m, m')^{1/2}.$$

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[1]That is $d\gamma(t) = \left\|\frac{d\gamma}{dt}(t)\right\| dt$.

if N is sufficiently large. The general case $k > 1$ follows by induction. □

Hence we have constructed a dynamics on the subset of adapted fields.

Let γ be a u-curve and X a vector field tangent to the center direction. The following Proposition is fundamental.

Proposition 3.1. *Suppose that there exists $C > 0$ with the following property: for every u-curve γ there exists a vector field X such that (γ, X) is an adapted field and for all $n \geq 0$ large*

$$\frac{I_n^{\gamma,X}}{n} > C.$$

Then the map f has a positive Lyapunov exponent greater than $C/2$ in the center direction at Lebesgue almost every point.

Proof. Consider the set B of regular points for which the map f does not have a positive Lyapunov exponent greater than $C/2$ in the center direction, and let us assume for the sake of contradiction that B has a density point b. For ϵ small let $\gamma^\epsilon = (\gamma_x^\epsilon, \gamma_y^\epsilon, \gamma_y^\epsilon, \gamma_w^\epsilon) : [-\epsilon, \epsilon] \to M$ be the curve tangent to E_f^u such that $|\frac{d\gamma_x}{dt}(t)| = 1\ \forall t$ and $\gamma^\epsilon(0) = b$, and denote by Leb the Lebesgue measure on its image.

By absolute continuity of the unstable foliation (see [Pes04]),

$$\frac{Leb(\gamma^\epsilon \cap B)}{Leb(\gamma^\epsilon)} \xrightarrow[\epsilon \to 0]{} 1$$

Take $\epsilon = 2\pi\lambda^{2Nk}$ with N large so that $Leb(\gamma^\epsilon \cap B^c) < \frac{C}{2\log 2N} Leb(\gamma^\epsilon)$: then the curve $f^k\gamma^\epsilon$ is a u-curve (by Prop. 2.1). Consider X_ϵ so $(f^k\gamma^\epsilon, f_*^k X_\epsilon)$ satisfies the hypothesis and let $\chi(m) = \limsup_{n\to\infty} \frac{\log \|d_m f^n(X_\epsilon)\|}{n}$. Observe that $\chi(m) < \frac{C}{2}$ if $m \in B$. By hypothesis, we obtain

$$\int_{\gamma^\epsilon} \chi d\gamma^\epsilon = \int_{f^k\gamma^\epsilon} \chi \circ f^{-k} \frac{1}{\left\|\frac{df^k\gamma^\epsilon}{dt}\right\|} d(f^k\gamma^\epsilon) = \lambda^{2Nk} \int_{f^k\gamma^\epsilon} \chi \circ f^{-k} d(f^k\gamma^\epsilon)$$

$$\geq \lambda^{2Nk} \limsup_{n\to\infty} \int_{f^k\gamma^\epsilon} \frac{\log \left\|d_m f^n(f_*^k X_\epsilon)\right\|}{n} d(f^k\gamma^\epsilon) \geq C|f^k\gamma^\epsilon|\lambda^{2Nk} = CLeb(\gamma^\epsilon)$$

On the other hand, by equation (6), $\chi \leq \log 2N$ and thus

$$\begin{aligned}\int_{\gamma^\epsilon} \chi d\gamma^\epsilon &= \int_{\gamma^\epsilon \cap B} \chi dLeb + \int_{\gamma^\epsilon \cap B^c} \chi dLeb \\ &\leq \frac{C}{2} Leb(\gamma^\epsilon) + \log 2N Leb(\gamma^\epsilon \cap B^c) < CLeb(\gamma^\epsilon)\end{aligned}$$

which gives the contradiction. □

We are led then to study the value of

$$E(\gamma, X) := \frac{1}{|\gamma|} \int_\gamma \log \|d_m f(X)\| \, d\gamma$$

for adapted fields (γ, X).

4. The basis in the Central Direction

The study of $E(\gamma, X)$ will be achieved by introducing a convenient basis of E^c. For $m = (x, y, z, w) \in M$ define

$$\Omega(m) = 2 + N \cos x.$$

One verifies directly that $d_m f|E^c$ is represented by the matrix

$$d_m f|E^c = \begin{pmatrix} \Omega(m) & -1 \\ 1 & 0 \end{pmatrix}$$

We consider the orthogonal basis of E^c_m given by

$$s_m = (1, \Omega(m)); \quad u_m = (\Omega(m), -1)$$

and verify that

$$d_m f(s_m) = (0,1), \quad d_m f(u_m) = (1+\Omega^2(m), \Omega(m)). \tag{9}$$

Now if X is a vector field tangent to E^c we can write

$$X_m = \frac{\cos\theta^X(m)}{\sqrt{1+\Omega^2(m)}} s_m + \frac{\sin\theta^X(m)}{\sqrt{1+\Omega^2(m)}} u_m, \tag{10}$$

where $\theta^X(m)$ is the angle $\angle(X_m, s_m)$. Hence

$$d_m f(X_m) = \left(\sin\theta^X(m)\sqrt{1+\Omega^2(m)}, \frac{\cos\theta^X(m) + \sin\theta^X(m)\Omega(m)}{\sqrt{1+\Omega^2(m)}} \right) \tag{11}$$

which in turn implies

$$\begin{aligned} \|d_m f(X_m)\| &\geq |\sin\theta^X(m)|\sqrt{1+\Omega^2(m)} \\ &\geq |\sin\theta^X(m)||2 + N\cos x| \\ &\geq N|\sin\theta^X(m)||\cos x| - 2 \end{aligned} \tag{12}$$

Definition 4.1. *For $0 \leq a < b \leq 1$, a* strip *$S[a,b]$ is a region of the form*

$$S[a,b] = \{m = (x,y,z,w) \in M : x \in [a,b]\}$$

The length *of the strip $S[a,b]$ is $l(S[a,b]) = b - a$.*

By a harmless abuse of language we also call strips to the union of two strips, and extend the concept of length accordingly.

The critical strip *is the strip*

$$Crit = S[b_1, b_2] \cup S[b_3, b_4]$$

where $0 < b_1 < \pi/2 < b_2 < b_3 < 3\pi/2 < b_4 < 2\pi$ are such that

$$\cos b_1 = \cos b_4 = \frac{1}{\sqrt{N}} \qquad \cos b_2 = \cos b_3 = -\frac{1}{\sqrt{N}}$$

One verifies that

$$\begin{aligned} -\frac{2}{\sqrt{N}} < b_1 - \frac{\pi}{2} < -\frac{1}{\sqrt{N}}, \quad \frac{1}{\sqrt{N}} < b_2 - \frac{\pi}{2} < \frac{2}{\sqrt{N}} \\ -\frac{2}{\sqrt{N}} < b_3 - \frac{3\pi}{2} < -\frac{1}{\sqrt{N}}, \quad \frac{1}{\sqrt{N}} < b_4 - \frac{3\pi}{2} < \frac{2}{\sqrt{N}} \end{aligned} \tag{13}$$

In particular, $l(Crit) \leq \frac{4}{\sqrt{N}}$.

Lemma 4.1. *Let X be a vector field tangent to the center bundle. For m outside the critical strip we have*

$$|\Omega(m)| > \sqrt{N} - 2, \quad \text{and} \quad \|df(X_m)\| \geq \sqrt{N}|\sin\theta^X(m)| - 2.$$

Proof. If $m \in Crit^c$ then $|N\cos m| \geq \sqrt{N}$ and the claim follows from equation (12). □

Let us define the following cone:

$$\chi := \mathbb{R} \cdot \{(1,n):\ |n| \leq \sqrt[4]{N}\} \subset \mathbb{R}^2.$$

Definition 4.2. *An adapted field* (γ, X) *is called* good *if*

$$\forall m \in \gamma, \quad X_m \in \chi,$$

otherwise it is called bad.

This Manicheistic dichotomy is suitable to evaluate the expectation of vector growth.

Proposition 4.1. *For* N *sufficiently large, for every adapted field* (γ, X)*, it holds:*

(1) $E(\gamma, X) \geq -\log 2N$.
(2) *Furthermore, if* (γ, X) *is good then* $E(\gamma, X) \geq \frac{1}{7}\log N$.

The following Lemma will be useful to prove the above proposition:

Lemma 4.2. *For* N *sufficiently large, for every good adapted field* (γ, X)*, it holds:*

$$|\sin\theta^X(m)| \geq \frac{1}{\sqrt[3]{N}} \quad \forall m \notin Crit.$$

Proof. We compute for m off the critical strip:

$$|\sin\theta^X(m)| \geq |\sin\angle(X_m, (0,1))| - |\angle(s_m,(0,1))| \geq \frac{1}{2\sqrt[4]{N}} - \arcsin\frac{1}{\sqrt{1+\Omega^2(m)}}$$

and conclude the claim by Lemma 4.1. □

Proposition 4.1. The first claim follows directly from the fact that $\left\|df^{-1}|E^c\right\| \leq 2N$. Assume that (γ, X) is good, then

$$|\gamma| \cdot E(\gamma, X) = \int_{\gamma\setminus Crit^c} \log\|d_m f(X)\|\, d\gamma + \int_{\gamma\cap Crit} \log\|d_m f(X)\|\, d\gamma \tag{14}$$

By Lemmas 4.1 and 4.2, outside of the critical strip

$$\|d_m f(X_m)\| \geq \sqrt[6]{N} - 2 \tag{15}$$

and hence for sufficiently large N

$$|\gamma| \cdot E(\gamma, X) \geq (1 - \frac{8}{\sqrt{N}})(\frac{1}{6}\log(N) - 2)|\gamma| - \frac{8}{\sqrt{N}}\log 2N|\gamma| \geq \frac{\log N|\gamma|}{7}$$

□

As we said, we continue working with N for which the previous results hold.

5. Transitions.

Now that we have concrete bounds for $E(\gamma, X)$, we want to understand the proportion of good fields obtained after iterating a given one. Recall that $f^k \circ \gamma = \gamma_1^k * \cdots \gamma_{[\mu^{2kN}]}^k * \gamma_{[\mu^{2kN}]+1}^k$. We define for every adapted field (γ, X):

$$G_k = G_k(\gamma, X) = \left\{1 \leq j \leq [\mu^{2kN}] : \left(\gamma_j^k, \frac{f_*^k X}{\|f_*^k X\|}\right) \text{ is Good}\right\}$$

$$B_k = B_k(\gamma, X) = \left\{1 \leq j \leq [\mu^{2kN}] : \left(\gamma_j^k, \frac{f_*^k X}{\|f_*^k X\|}\right) \text{ is Bad}\right\}$$

Lemma 5.1. *If* (γ, X) *is a good adapted field and* $f^{-1}\gamma_j^1 \cap Crit \neq \emptyset$*, then the field* $(\gamma_j^1, \frac{f_* X}{\|f_* X\|})$ *is good.*

Proof. Let $m \notin Crit$. For every $|n| \leq \sqrt[4]{N}$, we recall that

$$d_m f(1, n) = (\Omega(m) - n, 1).$$

As $|\Omega(m) - n| \geq |\Omega(m)| - |n| \geq \sqrt{N} - 2 - \sqrt[4]{N}$ by Lemma 4.1, it comes $d_m f(1, n) \in \chi$ and the lemma follows. □

Lemma 5.2. *For every N sufficiently large, for every bad adapted field (γ, X) there exists a strip S_X of length π so that for every j satisfying $f^{-1}\gamma_j^1 \subset S_X$, the field $(\gamma_j^1, \frac{f_* * X}{\|f_* * X\|})$ is good.*

Proof. For $m = (x, y, z, w)$ and $m_0 = (x_0, y_0, z_0, w_0)) \in M$, one has

$$df_m(X_m) = d_m f(X_m - X_{m_0}) + d_m f(X_{m_0}) \tag{16}$$

where by Corollary 3.1 the first vector on the right hand side has norm less than $2N\lambda^{N/3}$. Observe that $(s(m))_{m\in\gamma}$ takes all directions but the ones belonging to the vertical cone

$$\mathcal{E} = \mathbb{R} \cdot \{(1, n) : \ |n| \geq 2 + N\}$$

We consider two cases. First assume the existence of $m_0 \in \gamma$ such that X_{m_0} is colinear with $s(m_0)$. We compute

$$d_m f(s_{m_0}) = (\Omega(m) - \Omega(m_0), 1) = (N\cos(x) - N\cos(x_0), 1) \tag{17}$$

For every m with $x \in [x_0 + \pi - \frac{\pi}{2}, x_0 + \pi + \frac{\pi}{2}] + 2\pi\mathbb{Z}$ it holds

$$N|\cos x - \cos x_0| > N|\cos x_0| \tag{18}$$

As the adapted field X is bad and since X_{m_0} is colinear to $(1, N\cos x_0)$, we obtain

$$|N\cos x_0| > \sqrt[4]{N}. \tag{19}$$

By the previous equations, if $x \in [x_0 + \pi - \frac{\pi}{2}, x_0 + \pi + \frac{\pi}{2}]$ then the first component of $df_m(X_m)$ is bigger than $\sqrt[4]{N} - 2N\lambda^{N/3}$, and hence $df_m(X_m)$ makes a small angle with the x axis. We conclude that $df_m(X_m) \in \chi$ if $x \in [x_0 + \pi - \frac{\pi}{2}, x_0 + \pi + \frac{\pi}{2}]$.

In the second case, we assume that X_m is never colinear with s_m. As the image of $(s_m)_{m\in\gamma}$ is the whole cone χ, by the intermediate value theorem there exist some m_0 such that $X_{m_0} \notin \chi$.

In other words, X_{m_0} is colinear with a vector $(1, n)$ with $|n| \geq \sqrt[4]{N}$. For every $m \in \gamma$, the vector $df_m(X_{m_0})$ is colinear with the vector $(N\cos x + 2 - n, -1)$. Hence, for every m such that $\cos x$ is of the same sign as $-n$, the vector $df_m(X_{m_0})$ makes a small angle with the x axis, and thus is in χ. Such condition on the cosine corresponds to a strip of length π. □

These lemmas enable us to bound the ratio of transition between the number of bad and good adapted fields. Indeed, for every u-curve γ, the elements of the partition $(f^{-1}(\gamma_j^1))_{1\leq j\leq[\mu^{2N}]}$ have length very small and equivalent when N is large. Since the first coordinate of γ has constant derivative we can deduce the following proposition from the two above lemmas:

Proposition 5.1. *For every bad adapted field (γ, X), we have:*

$$\#G_1(\gamma, X) \geq \frac{\mu^{2N}}{3}, \quad \#B_1(\gamma, X) \leq \frac{2\mu^{2N}}{3} \quad \textit{with \# the cardinality of a set.}$$

For every good adapted field (γ, X), we have:

$$\#G_1(\gamma, X) \geq (1 - \frac{5}{2\pi\sqrt{N}})\mu^{2N}, \quad \#B_1(\gamma, X) \leq \frac{5}{2\pi\sqrt{N}}\mu^{2N}.$$

Proof. By the previous Proposition, if (γ, X) is bad, there exists a band S_X of length equal to π such that for every j satisfying $f^{-1}\gamma_j^1 \subset S_X$ the adapted field $(\gamma_j^1, \frac{(f_N)_* X}{\|(f_N)_* X\|})$ is good. This corresponds to almost half of curves γ_j^1. The second part is analogous, using $l(Crit) \leq \frac{4}{\sqrt{N}}$. □

Let $\eta = \eta_N := \frac{5}{2\pi\sqrt{N}}$. The Proposition permits us to readily calculate

$$\begin{aligned} \#G_{k+1} &\geq (1-\eta)\mu^{2N}\#G_k + \frac{1}{3}\mu^{2N}\#B_k \\ \#B_{k+1} &\leq \eta\mu^{2N}\#G_k + \frac{2}{3}\mu^{2N}\#B_k + \mu^{2N}. \end{aligned} \tag{20}$$

The last term $+\mu^{2N}$ is given by the possible bad curves comming from the slice $\gamma^{k+1}_{[\mu^{2N}]}$ of $f^{k+1}\gamma$ which is not a u-curve.

Lemma 5.3. *If N is sufficiently big then for all good adapted fields (γ, X) and for all $k \geq 1$ we have $\#G_k > 100\#B_k$.*

Proof. The proof is done by induction. By hypothesis $\#G_0 > 100\#B_0 = 0$. Obverse also that $\#G_1 > 100\#B_1 \neq 0$. Assume the claim was established for $k \geq 1$. Then by the previous Lemma and the induction hypothesis

$$\frac{\#B_{k+1}}{\#G_{k+1}} \leq \frac{\eta\#G_k + \frac{2}{3}\#B_k + 1}{(1-\eta)\#G_k + \frac{1}{3}\#B_k} \leq \frac{\eta\#G_k + \frac{2}{300}\#G_k + \mu^{2N}}{(1-\eta)\#G_k + \frac{1}{300}\#G_k} \leq \frac{\eta + \frac{2}{300}}{(1-\eta) + \frac{1}{300}} + \frac{1}{2\mu^{2N}}$$

The limit of the last term when N is smaller than $1/150 < 1/100$, hence for N sufficiently large (independently of k), $\#G_{k+1} > 100\#B_{k+1}$. □

Proof of Theorem 1.1 for f_N.

Take (γ, X) a good adapted field and recall that (8):

$$I_n^{\gamma,X} = \sum_{k=1}^{n-1} \frac{1}{\mu^{2Nk}|\gamma|} \left(\sum_{j=0}^{[\mu^{2kN}]} \int_{\gamma_j^k} \log \left\| d_m f(Y^k) \right\| d\gamma_j^k + \int_{\gamma^k_{[\mu^{2kN}]+1}} \log \left\| d_m f(Y^k) \right\| d\gamma^k_{[\mu^{2kN}]+1} \right) \tag{21}$$

Observe that the length of u-curves is almost constant by Corollary 2.1, and so $|\gamma_j^k| > \frac{9}{10}|\gamma|$. Note that by the previous Lemma $\#G_k > \frac{1}{1+\frac{1}{100}}[\mu^{2kN}]$. Splitting between bad and good adapted fields and using Proposition 4.1, we deduce[2]

$$\begin{aligned} \frac{I_n^\gamma}{n} &> \frac{1}{n}\sum_{k=0}^{n-1} \frac{9}{10\mu^{2Nk}} \left(\#G_k \frac{\log N}{7} - (\#B_k + 1)\log 2N \right) \\ &> \frac{1}{n}\sum_{k=0}^{n-1} \frac{9}{10} \frac{1}{1+\frac{1}{100}} \left(\frac{\log N}{7} - \frac{\log 2N}{100} - \frac{(1+\frac{1}{100})\log 2N}{\mu^{2kN}} \right) > \frac{1}{20}\log N \end{aligned} \tag{22}$$

An application of Proposition 3.1 finishes the proof. □

6. Robustness of Non-Uniform Hyperbolicity

In this section we indicate the relevant changes to our procedure to achieve the proof of Theorem 1.1, namely that there is a small neighbourhood $U \subset Diff^2_{leb}(M)$ of f_N formed by C^2-conservative diffeomorphisms having a NUH attractor with basin of full Lebesgue measure. We take N large such that the Main Theorem and its intermediate results hold for f_N.

The neighborhood U will be chosen small depending on N supposed.

We start by fixing U such that every $g \in U$ is partially hyperbolic, and

[2] Observe that for $k = 0, M_0 = 0$.

(A) for all unit vectors $v^s \in E^s_g$, $v^c \in E^c_g$ and $v^u \in E^u_g$, the following inequalities holds:

$$0.99\lambda^{2N} \leq \|dg(v^s)\| \leq 1.01\lambda^{2N} \ll 1$$
$$1 \ll 0.99\mu^{2N} \leq \|dg(v^u)\| \leq 1.01\mu^{2N}$$
$$\frac{1}{2N} \leq \|dg(v^c)\| \leq 2N.$$

(B) the following bounds hold $\left\|d^2g\right\|, \left\|d^2g^{-1}\right\| \leq 2N$.

(C) E^c_g is 1/2-Hölder.

We comment on the latter point. Given a partially hyperbolic map g consider constants $\varsigma, \upsilon, \varrho, \widehat{\varrho}, \widehat{\upsilon}, \widehat{\varsigma}$ such that for all unit vectors $v^s \in E^s_g$, $v^c \in E^c_g$ and $v^u \in E^u_g$:

$$\varsigma < \|d_m g(v^s)\| < \upsilon, \quad \varrho < \|d_m g(v^c)\| < \widehat{\varrho}^{-1}, \quad \widehat{\upsilon}^{-1} < \|d_m g(v^u)\| < \widehat{\varsigma}^{-1}$$

Theorem 6.1 (Pugh-Shub-Wilkinson [PSW12]). *If g is of class $\mathcal{C}^2$ and $\theta \in (0,1)$ satisfies*

$$\upsilon < \varrho\varsigma^\theta, \widehat{\upsilon} < \widehat{\varrho}\widehat{\varsigma}^\theta$$

then E^c_f is θ-Hölder.

Applying this theorem and using (A), we conclude that for N large enough and then U small enough, for every $g \in U$ the bundle E^c_g is 1/2-Hölder. Furthermore, since the center bundle depends continuously on the map, the Hölder constant K_g of E^c_g can be taken arbitrarily close to 1 $(= K_f)$.

Definition 6.1. *A* u-curve *for g is a curve $\gamma = (\gamma_x, \gamma_y, \gamma_z, \gamma_w) \colon [0, 2\pi] \to M$ tangent to E^u_g and such that $|\frac{d\gamma_x}{dt}(t)| = 1$, $\forall t \in [0, 2\pi]$. For every $k \geq 0$ there exists an integer $N_k = N_k(\gamma)$ such that the curve $g^k \circ \gamma$ can be written as a concatenation*

$$g^k \circ \gamma = \gamma^k_1 * \cdots \gamma^k_{N_k} * \gamma^k_{N_k+1} \tag{23}$$

where γ^k_j $j = 1, \ldots, N_k$ are u-curves and $\gamma^k_{N_k+1}$ is a segment of u-curve.

The unstable bundle also depends continuously on the perturbation, and thus we obtain the following analog to Corollary 2.1, for U small enough.

Corollary 6.1. *Let $m \in M$ and let $v = (v_x, v_y, v_z, v_w)$ be a unit vector in $E^u_g(m)$. Then*

$$\lambda^N(1 - 2\lambda^N) \|P_x(e^u)\| \leq |v_x| \leq \lambda^N(1 + 2\lambda^N) \|P_x(e^u)\|.$$

Every pair of u-curves (γ, γ') for g satisfy:

$$0.9 \cdot Leb(\gamma) \leq Leb(\gamma') \leq 1.1 \cdot Leb(\gamma)$$

Definition 6.2. *An adapted field (γ, X) for g is the pair of a u-curve γ and of a unit vector field X satisfying:*

1. *X is tangent to the center direction E^c_g.*
2. *X is $(C_X, 1/2)$-Hölder along γ, with $C_X < 10N^2\lambda^N$.*

The action of the map f on unstable leaves corresponds simply to multiplication by a constant factor. To deal with the perturbative case we need distortion bounds. Given a map $g \in U$ and a integer k we denote by $J^u_{g^k}$ the *unstable Jacobian* of g^k, namely

$$J^u_{g^k}(m) := |det(d_m g^k|_{E^u_g})|, \quad \forall m \in M$$

Note that from (A) follows

$$\forall m \in M, \quad \lambda^{2N}/1.01 \leq J^u_{g^{-1}}(m) \leq \lambda^{2N}/0.99 \tag{24}$$

Lemma 6.1 (Distortion bounds)**.** *There exists a constant $D = D_N$ with the following property. For all $g \in U$ and u-curve γ for g, for every $k \geq 0$, it holds*

$$\forall m, m' \in \gamma, \quad \frac{1}{D} \leq \frac{J^u_{g^{-k}}(m)}{J^u_{g^{-k}}(m')} \leq D$$

Furthermore, for U small in terms of N the number D can be taken close to 1.

This Lemma is classical. We present the proof nonetheless, since the version given is adapted to our purposes.

Proof. Fix $g \in U$ and u-curve γ for g. For $m, m' \in \gamma$ and an integer j, we denote by $m_j = g^j(m)$, $m'_j = g^j(m')$. We compute

$$\begin{aligned}
\log \frac{J^u_{g^{-k}}(m)}{J^u_{g^{-k}}(m')} &= \sum_{j=-k+1}^{0} \log \left| \frac{det(d_{m_j} g^{-1}|_{E^u_g})}{det(d_{m'_j} g^{-1}|_{E^u_g})} \right| \leq \sum_{j=-k}^{0} \left| \frac{det(d_{m_j} g^{-1}|_{E^u_g}) - det(d_{m'_j} g^{-1}|_{E^u_g})}{det(d_{m'_j} g^{-1}|_{E^u_g})} \right| \\
&\leq \frac{1.01}{\lambda^{2N}} \sum_{j=-k}^{0} \left| det(d_{m_j} g^{-1}|_{E^u_g}) - det(d_{m'_j} g^{-1}|_{E^u_g}) \right| \quad \text{(by equation (24))} \\
&\leq \frac{1.01}{\lambda^{2N}} \sum_{j=-k+1}^{0} C_0 d(m_j, m'_j) \leq \frac{1.01 C_0}{\lambda^{2N}} \sum_{j=-k+1}^{0} \left(\frac{\lambda^{2N}}{0.99} \right)^{|j|} d(m_0, m'_0)
\end{aligned}$$

where C_0 is the Lipschitz constant of $\det dg|_{E^u}$ restricted to γ. Note that C_0 is small when g is $\mathcal{C}^2$-close to f. Altogether we conclude

$$\log \frac{J^u_{g^{-k}}(m)}{J^u_{g^{-k}}(m')} \leq \frac{1.01 C_0}{\lambda^{2N}} \frac{1}{1 - \frac{\lambda^{2N}}{0.99}} d(m_0, m'_0)$$

and since the length of u-curves is bounded from above the claim follows. □

From the Lemma and using the change of variables Theorem we obtain.

Corollary 6.2. *For all $g \in U$ and u-curve γ for g, for every measurable set $A \subset \gamma$, for every $k \geq 0$ it holds*

$$\frac{1}{D} \frac{Leb(A)}{Leb(\gamma)} \leq \frac{Leb(g^{-k}A)}{Leb(g^{-k}\gamma)} \leq D \frac{Leb(A)}{Leb(\gamma)}$$

The proof of the existence of the pre-NUH attractor for the maps $g \in U$ is obtained by analysing the quantity

$$I^{\gamma,X}_n = \frac{1}{|\gamma|} \int_\gamma \log \|d_m g^n(X)\| \, d\gamma$$

where (γ, X) is an adapted field for g.

Proposition 6.1. *Suppose that there exists $C > 0$ with the following property: for every u-curve γ, there exists a vector field X such that (γ, X) is an adapted field for g and for all large $n \geq 0$*

$$\frac{I^{\gamma,X}_n}{n} > C.$$

Then the map g has a positive Lyapunov exponent greater than $C/2$ in the center direction at Lebesgue almost every point.

Proof. We refer the reader to the proof of Proposition 3.1 since the arguments are completely analogous. Consider the set B of regular points for which the map g does not have a positive Lyapunov exponent greater than $C/2$ in the center direction, and let us assume for the sake of contradiction that B has a density point b.

Choose the curve $\gamma^\epsilon : [-\epsilon, \epsilon] \to M$ as

$$\gamma^\epsilon(t) = g^{-k} \circ \beta_k(t)$$

where β_k is the u-curve for g satisfying $\beta_k(0) = g^k(b)$, and observe that the length $Leb(\gamma^\epsilon)$ is small for k large, although this curve is possibly not symmetric around b. If we consider the left and right segments of γ^ϵ with common boundary b, we see that by the distortion bounds Lemma that their quotient is close to 1. With this and the absolute continuity of $\mathcal{F}^u$, since b is density point of B, we conclude that for γ^ϵ small enough (or equivalently k sufficiently large)

$$\frac{Leb(\gamma^\epsilon \cap B^c)}{Leb(\gamma^\epsilon)} < \frac{C}{2D}.$$

Using again distortion bounds, we can write for any point $m^k \in g^k\gamma^\epsilon$

$$J^u_{g^{-k}}(m^k) \geq \frac{Leb(\gamma^\epsilon)}{DLeb(g^k\gamma^\epsilon)}$$

Let $\chi(m) = \limsup_n \frac{1}{n} \log \|dg^n \circ X \circ g^k(m)\|$, for $m \in \gamma^\epsilon$ and with X given by the Proposition hypothesis.

$$\Rightarrow \int_{\gamma^\epsilon} \chi d\gamma^\epsilon = \int_{g^k\gamma^\epsilon} \chi \circ g^{-k} J^u_{g^{-k}} d(g^k\gamma^\epsilon) \geq \frac{Leb(\gamma^\epsilon)}{DLeb(g^k\gamma^\epsilon)} \int_{g^k\gamma^\epsilon} \chi \circ g^{-k} d(g^k\gamma^\epsilon)$$
$$\geq \frac{Leb(\gamma^\epsilon)}{DLeb(g^k\gamma^\epsilon)} C \cdot Leb(g^k\gamma^\epsilon) \geq \frac{C}{D} Leb(\gamma^\epsilon)$$

The rest of the proof is identical to Proposition 3.1. □

We proceed as we did in the case of f and study the integral

$$E(\gamma, X) := \frac{1}{|\gamma|} \int_\gamma \log \|d_m g(X)\| \, d\gamma$$

where (γ, X) is an adapted field for g.

Denote by $\pi \colon \mathbb{R}^4 \to \mathbb{R}^2 \equiv E^c_f$ the canonical projection, and for a vector field X on a curve γ, let

$$\widetilde{X_m} := \frac{\pi(X_m)}{\|\pi(X_m)\|} \quad \forall m \in \gamma.$$

Definition 6.3. *An adapted field (γ, X) for g is good if for every $m \in \gamma$, $\widetilde{X}_m \in \chi$.*

Proposition 6.2. *For U small, for all $g \in U$ and (γ, X) adapted field for g, it holds:*

(1) $E(\gamma, X) \geq -\log 2N$.
(2) *If (γ, X) is good then $E(\gamma, X) \geq \frac{1}{7} \log N$.*

Proof. The first part follows directly from property (A). For the second part, equation (15) implies that for every $m \in Crit^c \cap \gamma$ the norm $\left\| d_m f(\widetilde{X}_m) \right\| \geq \sqrt[6]{N} - 2$. Since the center bundle depends continuously with respect to the map, if U is sufficiently small, X_m is close to $\widetilde{X}_m$ and $d_m g(X_m)$ is close to $d_m f(\widetilde{X}_m)$. Thus for every $m \in Crit^c$

$$\|d_m g(X_m)\| \geq \left\| d_m f(\tilde{X}_m) \right\| - 1 \geq \sqrt[6]{N} - 3$$

The rest of the proof is completely analogous to Proposition 4.1. □

Let (γ, X) be an adapted field for a map $g \in U$. For $k \geq 0$ consider the pairs $(\gamma_j^k, Y^k|\gamma_j^k)$, $1 \leq j \leq N_k$ with $Y^k = \frac{g_*^k X}{\|g_*^k X\|}$.

Lemma 6.2. *Every possible pair $(\gamma_j^k, Y^k|\gamma_j^k)$ is an adapted field.*

This follows exactly in the same fashion as in Lemma 3.1.

Observe that

$$I_n^{\gamma,X} = \sum_{k=0}^{n-1} \frac{1}{|\gamma|} \int_\gamma \log \left\| d_{g^k m} g(Y^k \circ g^k(m)) \right\| d\gamma = \sum_{k=0}^{n-1} \frac{1}{|\gamma|} \int_{g^k\gamma} \log \left\| d_m g Y^k \right\| J_{g^{-k}}^u d\gamma$$

Writing $g^k\gamma$ as the concatenation (23), it comes as for (21):

$$I_n^{\gamma,X} = \sum_{k=0}^{n-1} \left(R_k + \sum_{j=0}^{N_k} \frac{1}{|\gamma|} \int_{\gamma_j^k} \log \left\| d_m g(Y^k) \right\| J_{g^{-k}}^u d\gamma_j^k \right),$$

where $R_k = \frac{1}{|\gamma|} \int_{\gamma_{N_k+1}^k} \log \left\| d_m g(Y^k) \right\| J_{g^{-k}}^u d\gamma_{N_k+1}^k$. By Corollary 6.1, for every j, k:

$$\frac{|\gamma_j^k|}{|\gamma|} \geq 0.9 \quad \text{and} \quad \frac{|\gamma_{N_k+1}^k|}{|\gamma|} \leq 1.1$$

Hence

$$I_n^{\gamma,X} \geq \sum_{k=0}^{n-1} \left(R_k + 0.9 \sum_{j=0}^{N_k} \min_{\gamma_j^k} (J_{g^{-k}}^u \cdot E(\gamma_j^k, Y^k)) \right).$$

Also by *(A)*:

$$|R_k| \leq 1.1 \max_{\gamma_{N_k+1}^k} |J_{g^{-k}}^u| \cdot \log \|dg|E_c\| \leq 1.1 (0.99\mu^{2N})^{-k} \log(2N),$$

Which approaches 0 when k approaches infinity. Hence by Cesàro mean:

$$\frac{1}{n} \sum_{k=0}^{n-1} |R_k| \to 0, \ n \to \infty.$$

Using again the Cesàro mean, and Proposition 6.1, to show that the map g has a positive Lyapunov exponent in the center direction, it suffices to show that:

Proposition 6.3. *For U small enough, for every $g \in U$, every admissible field (γ, X) for g, every $k \geq 0$, it holds*

$$\sum_{j=0}^{N_k} \min_{\gamma_j^k} (J_{g^{-k}}^u \cdot E(\gamma_j^k, Y^k)) \geq \frac{\log N}{1000}.$$

We recall that by the symmetry property of f_N, the inverse of g has also the same form as g and so the same argument for g^{-1} implies the main result.

Proof of Proposition 6.3. This follows from a lemma based on the same dichotomy between Good and bad adapted fields for g. Let $G_k(\gamma, X)$ and $B_k(\gamma, X)$ be the corresponding subsets of $\{\gamma_j^k, j \leq N_k\}$. This begins with the following lemma proved below.

Lemma 6.3. *For U small enough, for every $g \in U$, for every good adapted field (γ, X) for g, for every $k \geq 0$ it holds*

(1) $\sum_{j \in G_k} \min_{\gamma_j^k} J_{g^{-k}}^u \geq \frac{1}{100}$.
(2) $\sum_{j \in G_k} \min_{\gamma_j^k} J_{g^{-k}}^u \geq 100 \sum_{j \in B_k} \max_{\gamma_j^k} J_{g^{-k}}^u$.

Indeed:

$$\sum_{j=0}^{N_k}\min_{\gamma_j^k}(J^u_{g^{-k}}\cdot E(\gamma_j^k,Y^k))=\sum_{j\in G_k}\min_{\gamma_j^k}(J^u_{g^{-k}}\cdot E(\gamma_j^k,Y^k))+\sum_{j\in B_k}\min_{\gamma_j^k}(J^u_{g^{-k}}\cdot E(\gamma_j^k,Y^k))$$

And by Proposition 6.2 and then Lemma 6.3:

$$\sum_{j=0}^{N_k}\min_{\gamma_j^k}(J^u_{g^{-k}}\cdot E(\gamma_j^k,Y^k))\geq\frac{logN}{7}\sum_{j\in G_k}\min_{\gamma_j^k}J^u_{g^{-k}}-log2N\sum_{j\in B_k}\max_{\gamma_j^k}J^u_{g^{-k}}$$

$$\geq(\frac{logN}{7}-\frac{log2N}{100})\sum_{j\in G_k}\min_{\gamma_j^k}J^u_{g^{-k}}\geq\frac{\frac{logN}{7}-\frac{log2N}{100}}{100}\geq\frac{\log N}{1000}$$

which is the claim of Proposition 6.3.

Let us show Lemma 6.3. We start with the following

Lemma 6.4. *For every adapted field (γ,X) for g. For every positive integer $k\geq 0$, it holds*

$$\frac{1}{2D}\leq\sum_{j\in G_k}\min_{\gamma_j^k}J^u_{g^{-k}}+\sum_{j\in B_k}\max_{\gamma_j^k}J^u_{g^{-k}}\leq 2D.$$

Proof. Form the distortion bounds Lemma 6.1 and Corollary 6.1, we get

$$1=\frac{1}{|\gamma|}\int_\gamma d\gamma=\frac{1}{|\gamma|}\sum_{k=1}^{N_k+1}\int_{\gamma_j^k}J^u_{g^{-j}}(m^{j,k})d\gamma_j^k\geq\frac{1}{D}\left(\sum_{j\in G_k}\frac{|\gamma_j^k|}{|\gamma|}\min_{\gamma_j^k}J^u_{g^{-k}}+\sum_{j\in B_k\cup\{N_k+1\}}\frac{|\gamma_j^k|}{|\gamma|}\max_{\gamma_j^k}J^u_{g^{-k}}\right)$$

$$\Rightarrow 1\geq\frac{0.9}{D}\left(\sum_{j\in G_k}\min_{\gamma_j^k}J^u_{g^{-k}}+\sum_{j\in B_k}\max_{\gamma_j^k}J^u_{g^{-k}}-1.1(1.01\lambda)^{2kN}\right)$$

and the first inequality follows. The second one is similar. □

As D is close to 1 when U is small, the latter lemma with the second statement of Lemma 6.3 imply the first statement of Lemma 6.3. Hence it remains only to show the second statement of Lemma 6.3. For this end, we study the transitions between good and bad adapted fields.

Lemma 6.5. *For U small, for every $g\in U$, every adapted field (γ,X)*

1. *If (γ,X) is a good adapted field and if j is so that $g^{-1}\gamma_j^1$ does not intersects the strip Crit of length $4/\sqrt{N}$, then the field $(\gamma_j^1,\frac{g_*X}{\|g_*X\|})$ is good.*
2. *If (γ,X) is bad, there exists a strip S_X of length π so that for every j satisfying $g^{-1}\gamma_j^1\subset S_X$, the field $(\gamma_1^j,\frac{g_*X}{\|g_*X\|})$ is good.*

Proof. First observe that $\widetilde{g_*X}$ is close to $\frac{g_*\widetilde{X}}{\|g_*\widetilde{X}\|}$ which is itself close to $\frac{f_*\widetilde{X}}{\|f_*\widetilde{X}\|}$, for U small and this uniformly among $g\in U$ and (γ,X) adapted field for g.

Assume that (γ_g,X) is good. In Lemma 5.1 we proved that for every $m\in\gamma_g\cap Crit^c$

$$\frac{(f_*\widetilde{X})_{fm}}{\left\|(f_*\widetilde{X})_{fm}\right\|}\in\chi$$

and is uniformly away from the boundary. Hence we conclude that for every point $m\in\gamma_g\cap Crit^c$, $(\widetilde{g_*X})_{gm}\in\chi$ and the first part follows. The second part is similar using Lemma 5.2.

□

We are now ready to prove the second statement of Lemma 6.3. The proof is by induction. The case $k=0$ follows by hypothesis. Assume the claim for $k \geq 0$. By the distorsion estimate:

$$D \sum_{j\in G_{k+1}} Leb(\gamma_j^{k+1}) \min_{\gamma_j^{k+1}} J^u_{g^{-k-1}} \geq \int_{\sqcup_{j\in G_{k+1}}\gamma_j^{k+1}} J^u_{g^{-k-1}} d(g^{k+1}\gamma)$$

$$\geq \int_{\sqcup_{j\in G_{k+1}}\gamma_j^{k+1}\cap g(\sqcup_{i\in G_k}\gamma_i^k)} J^u_{g^{-k-1}} d(g^{k+1}\gamma) \geq \sum_{i\in G_k}\int_{\sqcup_{j\in G_{k+1}}\gamma_j^{k+1}\cap g(\gamma_i^k)} J^u_{g^{-k-1}} d(g^{k+1}\gamma)$$

$$\geq \sum_{i\in G_k}\int_{g^{-1}(\sqcup_{j\in G_{k+1}}\gamma_j^{k+1})\cap\gamma_i^k} J^u_{g^{-k}} d(g^k\gamma) \geq \sum_{i\in G_k}\min_{\gamma_i^k} J^u_{g^{-k}} Leb(g^{-1}(\sqcup_{j\in G_{k+1}}\gamma_j^{k+1})\cap\gamma_i^k)$$

By using Lemma 6.5, with $\eta = \frac{5\sqrt{N}}{2\pi}$, and then Corollary 6.1

$$Leb(g^{-1}(\sqcup_{j\in G_{k+1}}\gamma_j^{k+1})\cap\gamma_i^k) \geq (1-\eta)Leb(\gamma_i^k) \geq 0.9(1-\eta)\max_j Leb(\gamma_j^{k+1})$$

Hence

$$\sum_{j\in G_{k+1}}\min_{\gamma_j^k} J^u_{g^{-k-1}} \geq \frac{0.9(1-\eta)}{D}\sum_{i\in\in G_k}\min_{\gamma_i^k} J^u_{g^{-k}} \tag{25}$$

Similarly:

$$\frac{1}{D}\sum_{j\in B_{k+1}}\max_{\gamma_j^{k+1}} J^u_{g^{-k-1}} \leq \int_{\sqcup_{j\in B_{k+1}}\gamma_j^{k+1}} J^u_{g^{-k-1}} d(g^{k+1}\gamma)$$

$$\leq \int_{\sqcup_{j\in B_{k+1}}\gamma_j^{k+1}\cap g(\sqcup_{i\in B_k}\gamma_i^k)} J^u_{g^{-k-1}} d(g^{k+1}\gamma) + \int_{\sqcup_{j\in B_{k+1}}\gamma_j^{k+1}\cap g(\sqcup_{i\in G_k}\gamma_i^k)} J^u_{g^{-k-1}} d(g^{k+1}\gamma) + \lambda^{N/2},$$

where the last term corresponds to the integral over to the bad curves coming from $\gamma^k_{N_k+1}$:

$$\int_{g(\gamma^k_{N_k+1})} J^u_{g^{-k-1}} d(g^{k+1}\gamma) \leq 1.01\mu^{2N} Leb(\gamma^k_{N_k+1})(1.01\mu^{2N})^{-k-1}$$

$$\leq Leb(\gamma^k_{N_k+1})(1.01\mu^{2N})^{-k} \leq \lambda^{-N}(1-2\lambda^N)^{-1}(1.01\mu^{2N})^{-k} \leq \lambda^{N/2}$$

$$\begin{aligned}\frac{1}{D}\sum_{j\in B_{k+1}}\max_{\gamma_j^{k+1}} J^u_{g^{-k-1}} &\leq \sum_{i\in G_k} D\min_{\gamma_i^k} J^u_{g^{-k}} Leb(g^{-1}(\sqcup_{j\in B_{k+1}}\gamma_j^{k+1})\cap\gamma_i^k)\\ &\quad+\sum_{i\in B_k}\max_{\gamma_i^k} J^u_{g^{-k}} Leb(g^{-1}(\sqcup_{j\in B_{k+1}}\gamma_j^{k+1})\cap\gamma_i^k) + \lambda^{N/2}\\ &\leq \sum_{i\in G_k} D\min_{\gamma_i^k} J^u_{g^{-k}}\eta Leb(\gamma_i^k) + \sum_{i\in B_k}\max_{\gamma_i^k} J^u_{g^{-k}}\frac{2}{3}Leb(\gamma_i^k) + \lambda^{N/2},\end{aligned}$$

hence:

$$\sum_{j\in B_{k+1}}\max_{\gamma_j^k} J^u_{g^{-k-1}} \leq 1.1D^2\eta(\sum_{i\in G_k}\min_{\gamma_i^k} J^u_{g^{-k}}) + \frac{2,2}{3}D(\sum_{i\in B_k}\max_{\gamma_i^k} J^u_{g^{-k}}) + \lambda^{N/2}D$$

By the induction hypothesis:

$$\sum_{j\in B_{k+1}}\max_{\gamma_j^k} J^u_{g^{-k-1}} \leq (1.1D^2\eta + \frac{2,2}{300}D)(\sum_{i\in G_k}\min_{\gamma_i^k} J^u_{g^{-k}}) + \lambda^{N/2}D \tag{26}$$

Using equations (25) and (26), on the first statement of Lemma 6.3 given by induction, we finally conclude:

$$\frac{\sum_{j\in B_{k+1}} \max_{\gamma_j^k} J^u_{g^{-k-1}}}{\sum_{j\in G_{k+1}} \min_{\gamma_j^k} J^u_{g^{-k-1}}} \le \frac{(1.1D^2\eta + \frac{2,2}{300}D)}{\frac{0.9(1-\eta)}{D}} + 100\lambda^{N/2}D$$

which is less than $1/100$ for η, $\lambda^{N/2}$ and $(D-1)$ small enough, which can be obtained by taking first N large and then U small. □

7. Concluding Remarks.

Exactly in the same fashion one can prove the robust non-uniform hyperbolicity of the map

$$f_r(m) = (S_r(x,y) + P_x \circ A^{[r]}(z,w), A^{[2r]}(z,w))$$

for sufficiently large parameter $r \in \mathbb{R}$.

Also for every linear endomorphism L of $\mathbb{R}^2$ such that $P_x \circ L(1,0) \neq 0$, and $k \in \mathbb{Z}$, the map

$$(x,y,z,w) \mapsto (S_N(x,y) + L(z,w), A^N(z,w))$$

preserves the two dimensional Lebesgue measure of the center bundle. Its perturbations are close to satisfy the same. Hence, instead of using the reversibility of the standard map, from such a property one deduces the existence of the contracting central Lyapunov exponent from the lower bound on the expending central Lyapunov exponent. Moreover, instead of considering the map S_N equal to the standard map, it seems also possible to perform the same method with the following class of maps:

$$(x,y) \mapsto (px - y + N\sin(x), x), \quad \text{with } p \in \mathbb{Z}$$

The presented method can also be used to give a new proof of the (ought to be) known robust non-uniform hyperbolicity of the map considered in [Shu71].

We can also wonder about the existence of a NUH attractor for non conservative perturbations of f_N. However very few works have been done on the existence of such physical measures when there are both positive and negative central Lyapunov exponents.

Let us point out that the map f_N can be $\mathcal{C}^2$-approximated by a stably ergodic diffeomorphism [SW00b]. It is possible that f_N is stably ergodic itself. By [BW10] it would suffice to prove accessibility and center bunching (see the aforementioned article for the definitions). Center bunching is immmediate, but accessibility seems harder to prove. In the previously known examples, a perturbation is made to guarantee accessibility. The use of Lemma 3.1 allows us to avoid this (or ergodicity) altogether.

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